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Monodromy of the matrix Schrödinger equations and Darboux transformations

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Abstract. A Schrödinger operator $L = -\mathrm{d}^2/\mathrm{d}z^2 + U(z)$ with a matrix-valued rational potential $U(z)$ is said to have trivial monodromy if all the solutions of the corresponding Schrödinger equations $L\psi = \lambda\psi$ are single-valued in the complex plane $z \in \mathbb{C}$ for any λ. A local criterion of this property in terms of the Laurent coefficients of the potential U near its singularities, which are assumed to be regular, is found. It is proved that any such operator with a potential vanishing at infinity can be obtained by a matrix analogue of the Darboux transformation from the Schrödinger operator $L_0 = -\mathrm{d}^2/\mathrm{d}z^2$. This generalizes the well known Duistermaat–Grünbaum result to the matrix case and gives the explicit description of the Schrödinger operators with trivial monodromy in this case.

1. Introduction

In 1986 Duistermaat and Grünbaum [1] proved the following result, which appears very classical but seems to have been unknown before.

Consider a Schrödinger operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + u(z) \tag{1}$$

with a rational potential $u(z)$ and the corresponding Schrödinger equation

$$L\psi = \lambda\psi \tag{2}$$

in the complex plane $z \in \mathbb{C}$.

Assume that all the singularities of $u(z)$ are regular, i.e. $u(z)$ has the poles of order at most 2. In general, this equation has a non-trivial monodromy around such a singularity depending on the spectral parameter λ. If this is not the case, i.e. all the solutions of the equation (2) are single-valued in the complex domain *for all* λ, we say that the corresponding Schrödinger operator has *trivial monodromy*.

Theorem ([1]). A Schrödinger operator L with a rational potential vanishing at ∞ has trivial monodromy if and only if L can be obtained from $L_0 = -\mathrm{d}^2/\mathrm{d}z^2$ by finitely many Darboux transformations.

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The corresponding potential $u(z)$ can be given by the following formula:

$$u = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log w(\psi_1, \psi_2, \dots, \psi_n) \tag{3}$$

where $w(\psi_1, \psi_2, \dots, \psi_n)$ is the Wronskian of the polynomial functions $\psi_1, \psi_2, \dots, \psi_n$ such that $\psi_1'' = 0$ and $\psi_{j+1}'' = \psi_j$ for $j = 1, 2, \dots, n-1$ and n is the number of Darboux transformations applied (see [2–4]) .

The Darboux transformation [5] of a Schrödinger operator L is defined in the following way. Let us first factorize L as

$$L = -(D+f)(D-f) \qquad D = \frac{\mathrm{d}}{\mathrm{d}z}.$$

To do this one can take any non-zero ψ from the kernel of L

$$L\psi = 0$$

and find f from $(D-f)\psi = 0$: $f = \psi' \cdot \psi^{-1} = (\log\psi)'$. Now we can define a new operator

$$\tilde{L} = -(D-f)(D+f).$$

Iterating this procedure n times applied to the given operator $L = L_0$ we arrive at an operator L_n which satisfies the relation

$$L_n A_n = A_n L_0 \tag{4}$$

where A_n is the differential operator of order n:

$$A_n = (D - f_{n-1})(D - f_{n-2}) \cdots (D - f_0). \tag{5}$$

In the case where $L_0 = -D^2$ the kernel of A_n is generated by the functions $\psi_1, \psi_2, \dots, \psi_n$ described above. In terms of these functions A_n can be written as

$$A_n(\phi) = \frac{w(\psi_1, \dots, \psi_n, \phi)}{w(\psi_1, \dots, \psi_n)}.$$

The aim of the present paper is to generalize these results into the matrix case and to describe the Schrödinger operators

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}z^2} + U(z) \tag{6}$$

with a matrix-valued rational potential $U(z)$ having trivial monodromy.

In section 2 we introduce the notion of the matrix Darboux transformation and give some formulae in terms of the so-called *quasideterminants* introduced by Gelfand and Retakh in [6].

In section 3 we find the local conditions on the potential of a Schrödinger operator L with trivial monodromy. We then use these conditions to prove our main result which is as follows.

Theorem. Let L be a matrix Schrödinger operator with a rational potential $U(z)$ vanishing at ∞. Assume that all its singularities are regular, i.e. $U(z)$ has the poles of order at most 2. Then L has trivial monodromy if and only if L is a result of matrix Darboux transformation applied to $L_0 = -\mathrm{d}^2/\mathrm{d}z^2$.

Although this result looks like a straightforward generalization of the Duistermaat–Grünbaum theorem, the proof is different even in the scalar case $d = 1$. It is based on the ideas of the recent paper by Chalykh on multidimensional scalar Schrödinger operators [7].

2. Matrix Darboux transformations and quasideterminants

Consider the Schrödinger operator

$$L = -D^2 + U(z) \qquad D = \frac{\mathrm{d}}{\mathrm{d}z} \tag{7}$$

with a potential $U(z)$ being $d \times d$ matrix-valued function.

Definition. We will say that the operator L is obtained from another operator $L_0 = -D^2 + U_0(z)$ by *matrix Darboux transformations* (MDT) if there exists a differential operator $A = D^n + a_1(z)D^{n-1} + \cdots + a_n(z)$ such that

$$LA = AL_0. \tag{8}$$

In other words, there exists a matrix differential operator A with the identity highest coefficient which intertwines L and L_0. The order n of A is called *the order of the MDT*.

The classical Darboux transformation [5] corresponds to the scalar case $d = 1$ and the order $n = 1$. Indeed, in this case one can prove (see, e.g., [8]) that $A = D - f$ where $f = (\log\psi)'$ for some eigenfunction ψ of L_0

$$L_0\psi = \lambda\psi.$$

This means that $L_0 - \lambda I$ can be factorized in the form

$$L_0 - \lambda I = -(D + f)(D - f)$$

and

$$L - \lambda I = -(D - f)(D + f).$$

In the matrix case with $d > 1$ this may not be true even if $n = 1$. So, the matrix Darboux transformation is in general not related to any factorization of $L_0 - \lambda I$ in contrast to the scalar case.

The following simple result explains the nature of the MDT (cf [8]).

Theorem 1. The kernel of the intertwining operator A is invariant under L_0

$$L_0(\mathrm{Ker}\, A) \subset \mathrm{Ker}\, A.$$

Conversely, for any nd-dimensional L_0-invariant subspace V of d-vector functions there exist the operators A and L such that $\mathrm{Ker}\, A = V$ and

$$LA = AL_0.$$

Proof. The first statement is obvious: from (8)

$$A(L_0\psi) = (LA)\psi = 0$$

if $A\psi = 0$. Thus $L_0(\mathrm{Ker}\, A) \subset \mathrm{Ker}\, A$. To prove the inverse statement we construct first the operator $A = D^n + a_1D^{n-1} + \cdots + a_n$ which has the given kernel V. This can be done in the same way as in the scalar case (see, e.g., [9]). Then we consider the operator AL_0. Since V is L_0-invariant $AL_0(V) = 0$, i.e. $V \subset \mathrm{Ker}\, AL_0$. This means the the operator AL_0 is right-divisible by A:

$$AL_0 = LA.$$

One can check easily that if L_0 is a Schrödinger operator then L has to be a Schrödinger operator as well.

To write down the explicit formulae for A and L we will use the notion of the *quasideterminant* introduced by Gelfand and Retakh [6].

Let $R = \mathrm{Mat}_d(\mathbb{C})$ be an algebra of matrices $d \times d$ and X be an $n \times n$ matrix over R (in [6] quasideterminants were introduced for any associative algebra R). For any $1 \leqslant i, j \leqslant n$ let $r_i(X)$ be the ith row and $c_j(X)$ be the jth column of X. Let X^{ij} be the submatrix of X obtained by removing the ith row and the jth column from X. For a row vector r let $r^{(j)}$ be r without the jth entry. For the column vector c let $c^{(i)}$ be c without the ith entry. Then the quasideterminant

$$|X|_{ij} = x_{ij} - r_i(X)^{(j)}(X^{ij})^{-1}c_j(X)^{(i)}$$

where x_{ij} is the ijth entry of X. If $d = 1$ then

$$|X|_{ij} = (-1)^{(i+j)}\frac{\det X}{\det X^{ij}}.$$

We should say that the term 'quasideterminant' may be misleading, since it corresponds to a generalization of the fraction of determinants of the matrix and its submatrix, but not to a determinant itself. Note also that the quasideterminant is not always defined, in contrast to the scalar case. It is easy to check the following properties (see [6]):

(a) the quasideterminant $|X|_{ij}$ does not change after the permutation of rows and columns in X provided the element x_{ij} is preserved;
(b) if the quasideterminant $|X|_{ij}$ of a matrix X is defined then $|X|_{ij} = 0$ is equivalent to the fact that the jth column of matrix X is a right linear combination of other columns of this matrix (columns are multiplied by the elements of R from the right).

Let us combine the vectors of a basis in L_0-invariant space V as a columns of n $d \times d$ matrices $\Psi_1, \dots, \Psi_n$: $V = \langle \Psi_1, \dots, \Psi_n \rangle$. In terms of the matrices $\Psi_1, \dots, \Psi_n$ the intertwining operator A can be written as

$$A(\Psi) = |W(\Psi_1, \dots, \Psi_n, \Psi)|_{n+1,n+1} \tag{9}$$

where

$$W(\Psi_1, \dots, \Psi_n, \Psi) = \begin{pmatrix} \Psi_1 & \cdots & \Psi_n & \Psi \\ \vdots & \ddots & \vdots & \vdots \\ \Psi_1^{(n-1)} & \cdots & \Psi_n^{(n-1)} & \Psi^{(n-1)} \\ \Psi_1^{(n)} & \cdots & \Psi_n^{(n)} & \Psi^{(n)} \end{pmatrix}$$

(see [9]). Then the potential $U(z)$ can be written as

$$U = U_0 + 2a_1'(z) \tag{10}$$

where $a_1(z)$ is the first matrix coefficient of A: $A = D^n + a_1(z)D^{n-1} + \cdots + a_n(z)$. If we assume that the subspace generated by the columns of the first $(n-1)$ matrices $\Psi_1, \dots, \Psi_{n-1}$ is L_0-invariant then one can write the following formula for the potential U of the operator L (cf [9])

$$U = U_0 - 2(Y_{nn}W_{nn}^{-1})' \tag{11}$$

where

$$W = W(\Psi_1, \dots, \Psi_n) = \begin{pmatrix} \Psi_1 & \cdots & \Psi_n \\ \vdots & \ddots & \vdots \\ \Psi_1^{(n-2)} & \cdots & \Psi_n^{(n-2)} \\ \Psi_1^{(n-1)} & \cdots & \Psi_n^{(n-1)} \end{pmatrix}$$

$$Y = Y(\Psi_1, \ldots, \Psi_n) = \begin{pmatrix} \Psi_1 & \cdots & \Psi_n \\ \vdots & \ddots & \vdots \\ \Psi_1^{(n-2)} & \cdots & \Psi_n^{(n-2)} \\ \Psi_1^{(n)} & \cdots & \Psi_n^{(n)} \end{pmatrix}.$$

and $Y_{nn} = |Y|_{n,n}$, $W_{nn} = |W|_{n,n}$.

In the special case $L_0 = -D^2$ any L_0-invariant space V is generated by the columns of the $d \times nd$ matrix Φ satisfying the equation

$$\Phi'' = \Phi C \tag{12}$$

where a constant $nd \times nd$ matrix C can be assumed to be in Jordan form:

$$C = \begin{pmatrix} J_1(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_2(\lambda_2) & & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & J_m(\lambda_m) \end{pmatrix} \qquad J(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

Note that in this case V consists of vectors with the quasipolynomial coordinates $x_j = \sum_{i=1}^{m} p_{ij} e^{\lambda_i z}$, $p_{ij}(z)$ are polynomials in z. Pure polynomial case corresponds to $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. Equation (11) gives the general form of the potential in this case where $\Phi = (\Psi_1, \ldots, \Psi_n)$ is any solution of the equation (12) with linearly independent columns.

In order to obtain the rational potential one has to consider the matrix C with all $\lambda_i = 0$. As follows from (9), (11) the singularities of the potential are the zeros of the (usual) determinant of the matrix $W = W(\Psi_1, \ldots, \Psi_n)$

$$\det W(z) = 0 \tag{13}$$

which in this case is polynomial on z. It is easy to show that the degree of this polynomial is $N_d(n) = nd(nd+1)/2$ (cf [10], where $d = 1$ is considered) in the case when C consists of only one Jordan block

$$C = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For other C the degree of $\det W$ is less then $N_d(n)$. Let us consider the generic case, when $\deg \det W = N_d(n)$ and all the roots of the equation (13) are simple. Then from the formulas (11), (12) one can derive that the potential $U(z)$ has the form

$$U(z) = \sum_{i=1}^{N_d(n)} \frac{A_i}{(z - z_i)^2} \tag{14}$$

where all the matrices A_i have rank 1.

3. Matrix Schrödinger operator with trivial monodromy

Let us consider the matrix Schrödinger equation

$$-\frac{d^2}{dz^2}\psi + U(z)\psi = E\psi \tag{15}$$

with a meromorphic potential $U(z)$. Let $z = z_0$ be a singularity of U. Without loss of generality we can consider the case $z_0 = 0$. We assume that this singularity is regular (see the definition in [11]), i.e. in the neighbourghood of zero $U(z)$ has an expansion of the form

$$U(z) = \frac{C_{-2}}{z^2} + \frac{C_{-1}}{z} + C_0 + C_1 z + \cdots = \sum_{r\geqslant -2}^{\infty} C_r z^r \tag{16}$$

where C_i are some constant $d \times d$ matrices.

Lemma 1. If the matrix Schrödinger equation (15), (16) for some $E \in \mathbb{C}$ has a complete basis of solutions which are meromorphic near $z = 0$ then C_{-2} is diagonalizable with the eigenvalues $\lambda_i = m_i(m_i - 1)$, $m_i \in \mathbb{Z}_+$.

The proof follows from analysis of the series expansions

$$\psi = z^\alpha(\psi_0 + z\psi_1 + \cdots + z^k\psi_k + \cdots) \tag{17}$$

satisfying (15), (16).

Thus V can be represented as a direct sum of the eigenspaces of C_{-2}:

$$V = \bigoplus_{m=1}^{M} V_m \qquad \dim V_m = d_m \qquad \sum_{m=1}^{M} d_m = d \tag{18}$$

where the eigenspace V_m corresponds to the eigenvalue $\lambda = m(m-1)$ (some of these spaces can have dimension $d_m = 0$).

Any operator in V

$$\mathbf{A} : \bigoplus_{i=1}^{M} V_i \to \bigoplus_{i=1}^{M} V_i$$

can be represented in a block form:

$$\mathbf{A} = (\mathbf{A}^{ij}) \qquad i, j = 1, \ldots, M$$

where $\mathbf{A}^{ij}$ are some operators

$$\mathbf{A}^{ij} : V_i \to V_j.$$

This corresponds to the representation of the matrix A of such an operator in a suitable basis in the block form

$$A = (A^{ij}) \qquad 1 \leqslant i, j \leqslant M \tag{19}$$

where A^{ij} are $d_i \times d_j$ matrices.

Any vector ψ in V can be uniquely represented as a sum

$$\psi = \sum_{i=1}^{M} \psi^i \qquad \psi^i \in V_i. \tag{20}$$

Now let us consider the case when all the solutions of the Schrödinger equation (15) are meromorphic *for all* $E \in \mathbb{C}$, i.e. the corresponding Schrödinger operator has trivial monodromy.

The following local criteria for this can be proved in the same way as in the scalar case [1]. We will consider all the matrix coefficients C_l and ψ_s from (16), (17) to be represented in the form (19), (20) related to the eigensplitting (18) of the coefficient C_{-2}. In particular, we assume that C_{-2} has the following diagonal form:

$$C_{-2} = \begin{pmatrix} 0\cdot I_1 & & & & & \\ & 2\cdot I_2 & & & & \\ & & \ddots & & & \\ & & & m(m-1)\cdot I_m & & \\ & & & & \ddots & \\ & & & & & M(M-1)\cdot I_M \end{pmatrix} \tag{21}$$

where I_m is $d_m \times d_m$ identity matrix. If some of $d_m = 0$ the corresponding entries in matrix coefficients should be omitted. We should note also that, in general, the splittings (18) depend on the singularity $z = z_0$.

Theorem 2. A matrix Schrödinger operator L with a meromorphic potential $U(z)$ has trivial monodromy if and only if the following entries of the corresponding matrix coefficients in the expansions of $U(z)$ near any of its singular points vanish:

$$\begin{aligned} &1. \qquad C_l^{ij} = 0 \qquad \text{if } \ |i-j| \geqslant l+1 \\ &2. \qquad C_l^{ij} = 0 \qquad \text{if } \ i+j = l+3, l+5, \ldots, l+2k+1, \ldots \end{aligned} \tag{22}$$

where $l = -1, 0, \ldots, 2M-3$. In particular, the matrix residue $C_{-1} = 0$.

The coefficients $\psi_0, \psi_1, \ldots, \psi_{2M-3}$ of the corresponding expansions of the vector-eigenfunctions

$$\psi = z^{-M+1}(\psi_0 + z\psi_1 + \cdots + z^k\psi_k + \cdots)$$

satisfy the conditions

$$\begin{aligned} &1. \qquad \psi_l^i = 0 \qquad \text{if } \ i+l < M \\ &2. \qquad \psi_l^i = 0 \qquad \text{if } \ i+l = M+1, M+3, \ldots, M+2k+1, \ldots \\ &\qquad\qquad\qquad\quad \text{and } \ l-i \leqslant M-3. \end{aligned} \tag{23}$$

The structure of the corresponding coefficients is shown below:

$$C_{2M-3} = \begin{pmatrix} \star & \cdots & & \star \\ & & & \\ \vdots & \ddots & & \vdots \\ & & & \star \\ \star & \cdots & \star & 0 \end{pmatrix}, \quad C_{2M-4} = \begin{pmatrix} \star & & \cdots & & \star \\ & & & & \vdots \\ \vdots & & \ddots & & \star \\ & & & \star & 0 \\ \star & \cdots & \star & 0 & \star \end{pmatrix},$$

$$C_{2M-5} = \begin{pmatrix} \star & & \cdots & & & \star \\ & & & & & \vdots \\ \vdots & & & & & \star \\ \vdots & & & \ddots & \star & 0 \\ & & & \star & 0 & \star \\ \star & \cdots & \star & 0 & \star & 0 \end{pmatrix}, \quad \ldots, \quad C_{M-2} = \begin{pmatrix} \star & & \cdots & & \star & 0 \\ & \ddots & & \star & 0 & \star \\ \vdots & & \star & 0 & \star & \vdots \\ & & \cdots & & & \\ \star & 0 & \star & & & \vdots \\ 0 & \star & 0 & & \cdots & \end{pmatrix}$$

and

$$C_k = \begin{pmatrix} \star & \cdots & \star & 0 & \cdots & 0 & 0 \\ \vdots & \star & 0 & \star & \ddots & & 0 \\ \star & 0 & & \ddots & \ddots & \ddots & \vdots \\ 0 & \star & \ddots & \ddots & \ddots & \star & 0 \\ \vdots & \ddots & \ddots & \ddots & & 0 & \star \\ 0 & & \ddots & \star & 0 & \star & \\ 0 & 0 & \cdots & 0 & \star & & \end{pmatrix} \qquad k < M - 2.$$

In particular

$$C_0 = \begin{pmatrix} \star & 0 & \cdots & 0 & 0 \\ 0 & \star & & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \star \end{pmatrix}.$$

Then

$$\psi_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \star \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \star \\ 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \star \\ 0 \\ \star \end{pmatrix}$$

and so on.

$$\psi_{M-1} = \begin{pmatrix} \star \\ 0 \\ \star \\ 0 \\ \star \\ \vdots \end{pmatrix}, \quad \psi_M = \begin{pmatrix} \star \\ \star \\ 0 \\ \star \\ 0 \\ \vdots \end{pmatrix}, \quad \ldots, \quad \psi_{2M-3} = \begin{pmatrix} \star \\ \star \\ \vdots \\ \star \\ \star \\ 0 \end{pmatrix}.$$

Consider now the main case when $U(z)$ is rational and decays at infinity. Since all the residues must be zero $U(z)$ can be represented as

$$U(z) = \sum_{i=1}^{N} \frac{A_i}{(z - z_i)^2}. \tag{24}$$

Relations (22) give the algebraic system for the matrix coefficients A_i and poles z_i which we will call as *locus equations*. This terminology goes back to the paper by Airault *et al* [10], who considered this system in the scalar case.

Let us consider the case (which is generic in some sense, see equation (14) in section 2) when the rank of all matrices A_i is 1 and the only non-zero eigenvalue of A_i is 2. Such matrices can be represented as

$$A_i = 2a_i \otimes b_i$$

where $a_i \in V^*$ is a covector, $b_i \in V$ is a vector such that $a_i(b_i) = 1$. By definition, for any $x \in V$

$$a_i \otimes b_i(x) = a_i(x)b_i.$$

Here the vector b_i is the eigenvector of A_i with eigenvalue 2:

$$A_i(b_i) = 2a_i(b_i)b_i = 2b_i$$

and the covector a_i defines the kernel of A_i by the relation $a_i(x) = 0$.

For the potential

$$U(z) = \sum_{i=1}^{N} \frac{2a_i \otimes b_i}{(z - z_i)^2} \tag{25}$$

the locus equations (22) can be written in the following explicit form:

$$\begin{aligned} &\left[a_i \otimes b_i, \sum_{j \neq i} \frac{a_j \otimes b_j}{(z_j - z_i)^2} \right] = 0 \qquad i = 1, \ldots, N \\ &\sum_{j \neq i} \frac{a_i(b_j)\, a_j(b_i)}{(z_j - z_i)^3} = 0 \qquad i = 1, \ldots, N. \end{aligned} \tag{26}$$

They coincide with the stationary equations for the matrix version of the Calogero–Moser system suggested by Gibbons and Hermsen [12] (see also [13]). The matrix Darboux transformation allows us to construct some solutions to this complicated algebraic system because of the following result.

Theorem 3. All matrix Schrödinger operators L obtained by Darboux transformation from $L_0 = -\mathrm{d}^2/\mathrm{d}z^2$ have trivial monodromy.

Proof. Indeed, the kernel of the intertwining operator A is invariant under L_0 and, therefore, is generated by linear combination of exponents and polynomials. This implies that all the coefficients of A are meromorphic in $\mathbb{C}$ and, therefore, any eigenfunction ψ of L can be written on the form

$$\psi = A\left(\mathrm{e}^{kz} R_1 + \mathrm{e}^{-kz} R_2\right)$$

where R_1 and R_2 are constant matrices.

It turns out that in such a way all the Schrödinger operators with trivial monodromy can be constructed. This follows from our main theorem:

Theorem 4. Let L be a matrix Schrödinger operator with a rational potential $U(z)$ vanishing at ∞. Suppose that the corresponding Schrödinger equation $L\psi = \lambda\psi$ has only regular singular points and L has trivial monodromy. Then L can be obtained by matrix Darboux transformation from $L_0 = -\mathrm{d}^2/\mathrm{d}z^2$.

Proof. We borrow the main idea from the recent paper by Chalykh [7]. Let

$$U(z) = \sum_{s=1}^{N} \frac{A_s}{(z - z_s)^2} \tag{27}$$

and $\lambda_s = M_s(M_s - 1)$ be the maximum of eigenvalues of the matrix A_s. Introduce a linear space V consisting of the vector functions $\psi(z),\ z \in \mathbb{C}$ satisfying the following conditions:

1. $\psi(z) \prod_{s=1}^{N} (z - z_s)^{M_s - 1}$ is holomorphic in $\mathbb{C}$.
2. The coefficients of the Laurent expansion of $\psi(z)$ at the vicinity of $z_s,\ s = 1, \ldots N$ satisfies the conditions (23) with $M = M_s$.

Due to theorem 2 the coefficients of the potential satisfy relations (22). One can check that this implies that the space V is invariant with respect to L (cf [7]).

Let us consider the vector function $\psi_0 = \prod_{s=1}^{N}(z-z_s)^{M_s-1}\mathrm{e}^{kz}v$, where v is a constant vector. Evidently, $\psi_0 \in V$ and, therefore, all vector functions

$$\psi_i = (L+k^2)^i\psi_0$$

belong to V as well. These functions have the form

$$\psi_i = P_i(k,z)\mathrm{e}^{kz}v$$

where $P_i(k,z)$ is a polynomial in k and a rational function in z. Since

$$P_{i+1} = \left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 2k\frac{\mathrm{d}}{\mathrm{d}z} + U(z)\right)P_i$$

the degree of P_i in z at infinity decreases with i: $\deg_z P_i \leqslant M - i$, $M = \sum_{s=1}^{N}(M_s-1)$. On the other hand, since the space V is invariant under L the degree of the denominator of P_i cannot be more than that of $\prod_{s=1}^{N}(z-z_s)^{M_s-1}$. So, there exists a K such that $(L+k^2)\psi_K = 0$. It is easy to see that for M defined above

$$\psi_M = \left[2^M M!k^{\sum_{s=1}^{N}(M_s-1)} + \cdots\right]\mathrm{e}^{kz}v \neq 0. \tag{28}$$

We claim that $\psi_{M+1} = (L+k^2)\psi_M = 0$. Indeed, assume that this is not true. Then for some $K > M$ $\psi_K \neq 0$,

$$\psi_{K+1} = (L+k^2)\psi_K = 0.$$

Since

$$P_{K+1} = \left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} - 2k\frac{\mathrm{d}}{\mathrm{d}z} + U(z)\right)P_K = 0$$

and P_K is polynomial in k its highest coefficient has to be constant. At the same time P_K has to decay at infinity at least as z^{M-K}. Thus $K = M$ and

$$L\psi_M = -k^2\psi_M.$$

Now consider v to be the basis vectors $e_1, e_2, \ldots, e_d$ and arrange the matrix Ψ with the corresponding vector functions Ψ_M as the columns. Replacing k by $\mathrm{d}/\mathrm{d}z$ we can define a differential operator A such that $\psi = A\mathrm{e}^{kz}I$. We have

$$L\Psi = LA\mathrm{e}^{kz}I = -k^2A\mathrm{e}^{kz}I = -Ak^2\mathrm{e}^{kz}I = AL_0\mathrm{e}^{kz}I$$

and, therefore,

$$LA = AL_0. \tag{29}$$

Thus, L is related to $L_0 = -\mathrm{d}^2/\mathrm{d}z^2$ by a matrix Darboux transformation. The theorem is proved.

Remark. We should mention that in contrast to the scalar case MDT in general does not preserve the regularity of the singular points. As an example one can take L_0 as above and

$$\Psi = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}.$$

After the corresponding matrix Darboux transformation (11) we have

$$U = \begin{pmatrix} -\frac{1}{z^2} & \frac{2}{z^3} \\ 0 & -\frac{1}{z^2} \end{pmatrix}$$

so $U(z)$ has the pole of order three at $z = 0$. Of course, this may happen only as a degenerate case when some of the singularities collide (cf [14]).

As a corollary to theorem 4 we give the following description of the Schrödinger operators with trivial monodromy in the simplest case when $d = 2$ and the potential $U(z)$ has only three second-order poles. From the explicit equations (11) and (12) it follows that these three poles can be arbitrary complex numbers, say u, v and w. They are the only essential parameters of the locus in this case: after a suitable transformation $U \to CUC^{-1}$ the potential $U(z)$ has the following form:

$$U(z) = \frac{P_u}{(z-u)^2} + \frac{P_v}{(z-v)^2} + \frac{P_w}{(z-w)^2}$$

where the projectors P_u, P_v, P_w are defined as follows:

$$P_u = \frac{2}{(u-w)(u-v)} \begin{pmatrix} wv - u^2 & u(u^2 - vw) \\ w - 2u + v & -u(w - 2u + v) \end{pmatrix}$$

$$P_v = \frac{2}{(v-w)(v-u)} \begin{pmatrix} uw - u^2 & v(v^2 - uw) \\ u - 2v + w & -v(u - 2v + w) \end{pmatrix}$$

and

$$P_w = \frac{2}{(w-u)(w-v)} \begin{pmatrix} uv - w^2 & w(w^2 - uv) \\ u - 2w + v & -w(u - 2w + v) \end{pmatrix}.$$

Note that this potential is symmetric $U = U^{\mathrm{T}}$ iff u, v and w are the three roots of the equation

$$z^3 + 3z + \tau = 0$$

for some $\tau \in \mathbb{C}$.

The general investigation of the symmetry and reality conditions as well as the spectral properties of the corresponding Schrödinger operators is still to be done. We should mention in this context the papers by Wadati [15] and Calogero and Degasperis [16], where some of these problems have been discussed in the relation to the matrix KdV equation (see also Agranovich and Marchenko [17]).

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